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LETTER TO THE EDITOR

Scaling theory of dead-end distribution in percolation clusters

T Ohtsuki and T Keyes
Department of Chemistry, Boston University, Boston, MA 02215, USA

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Abstract. On the basis of scaling assumptions, critical behaviour of distribution (number) of dead (dangling) ends in the infinite cluster of percolating systems is investigated. Critical exponents describing dead-end distribution are introduced and scaling relations between them are derived. The exponents are also expressed in terms of other exponents ν, β and β_B for percolation and their explicit values are evaluated from known estimates for ν, β and β_B.

In the last decade, there has been a growing interest in the study of percolation as a model describing essential features of random systems (for reviews, see Stauffer 1979, Essam 1980). The understanding of the geometrical structure of percolation clusters is indispensable in order to investigate physical properties such as conductivity, diffusivity and magnetic order. Near the percolation threshold p_c and within the length scale less than coherence length ξ, clusters are self-similar (Kapitulnik *et al* 1983) and various scaling relations hold (Stauffer 1979). Above p_c, the system consists of one infinite cluster and many finite clusters. The infinite clusters are also divided into two parts, the backbone and dead (dangling) ends (Kirkpatrick 1978). On the basis of the scaling assumption, the distribution (number) of finite clusters is well understood (Stauffer 1979), whereas little is known about the distribution (number) of dead ends. Since the infinite cluster reflects connectivity of the system and is responsible for most of the physical quantities, the investigation of dead-end distribution, which is the purpose of this letter, is of particular significance. Here we adopt scaling arguments similar to those for finite cluster distribution presented by Stauffer (1979).

In this article, we define a dead (dangling) end as follows. Consider the cluster connected to the backbone only at one root point. When a potential difference is applied between the root point and another point in the cluster, current flow is confined to part of the cluster. We define an s-sites dead end by the largest component of the cluster composed of s sites (bonds) which carry current. The remaining part of the cluster consists of some clusters attached to the dead ends at one point. For each cluster, an s'-sites dead end is defined in a similar way. Iterating this procedure until all sites in the infinite cluster are assigned to the backbone and dead ends, we can define the distribution of s-sites dead ends uniquely. This definition is different from the customary one where a dead end represents the whole cluster directly connected to the backbone.

Let $n_D(s, p)$ be a distribution function of s-sites dead ends defined by

$$n_D(s, p) = \frac{\text{number of } s\text{-sites dead ends}}{\text{number of sites belonging to the whole infinite cluster}}. \quad (1)$$

As usual (Stauffer 1979), we introduce critical exponents for dead-end distribution as

$$\left[\sum_s n_{\rm D}(s,p)\right]_{\rm sing} \propto (p-p_{\rm c})^{2-\alpha_{\rm D}}, \tag{2}$$

$$\left[\sum_s s n_{\rm D}(s,p)\right]_{\rm sing} \propto (p-p_{\rm c})^{\beta_{\rm D}}, \tag{3}$$

$$\left[\sum_s s^2 n_{\rm D}(s,p)\right]_{\rm sing} \propto (p-p_{\rm c})^{-\gamma_{\rm D}}, \tag{4}$$

$$\left[\sum_s s n_{\rm D}(s,p) e^{-hs}\right]_{\rm sing} \propto h^{1/\delta_{\rm D}}, \tag{5}$$

where Σ_s denotes the sum over all dead ends and the subscript 'sing' means the singular part, i.e., the leading non-analytic part of the subscripted quantity. By definition, we have

$$P_{\rm B} + \sum_s s n_{\rm D} = 1, \tag{6}$$

where $P_{\rm B}$ is the probability of sites in the infinite cluster belonging to the backbone. In the vicinity of the percolation threshold, $P_{\rm B}$ varies singularly as $P_{\rm B} \propto \xi^{d_{\rm B}}/\xi^{d_{\rm C}} \simeq (p-p_{\rm c})^{\beta_{\rm B}-\beta}$, where $d_{\rm B} = d - \beta_{\rm B}/\nu$ is the fractal dimensionality of the backbone and $d_{\rm C} = d - \beta/\nu$ is that of the whole cluster (Kirkpatrick 1978). Then we find $[\Sigma_s s n_{\rm D}]_{\rm sing} \simeq P_{\rm B}$ and

$$\beta_{\rm D} = \beta_{\rm B} - \beta. \tag{7}$$

We make scaling arguments in a normal way (Stauffer 1979). That is, we assume that all singular behaviour of dead-end distribution is dominated by an essentially unique typical dead-end size given by

$$s_\xi \propto (p-p_{\rm c})^{-1/\sigma_{\rm D}} \tag{8}$$

and the distribution function $n_{\rm D}$ is expressed in the scaling form

$$n_{\rm D}(s,p) = s^{-\tau_{\rm D}} F(s/s_\xi). \tag{9}$$

Substitution of equations (8) and (9) into equations (2)–(5) leads to scaling relations

$$2-\alpha_{\rm D} = (\tau_{\rm D}-1)/\sigma_{\rm D}, \qquad \beta_{\rm D} = (\tau_{\rm D}-2)/\sigma_{\rm D}, \qquad -\gamma_{\rm D} = (\tau_{\rm D}-3)/\sigma_{\rm D},$$
$$1/\delta_{\rm D} = \tau_{\rm D} - 2, \tag{10}$$

because $[\Sigma_s s^k n_{\rm D}]_{\rm sing} \propto (p-p_{\rm c})^{(\tau_{\rm D}-1-k)/\sigma_{\rm D}}$ and $[\Sigma_s s n_{\rm D}\, {\rm e}^{-hs}]_{\rm sing} \propto h^{\tau_{\rm D}-2}$ (for details, see Stauffer 1979). The hyperscaling relation is also derived on the basis of the scaling assumption about the radius R_s of an s-sites dead end. Here we postulate

$$R_s = \xi G(s/s_\xi). \tag{11}$$

On the other hand, R_s is related to s by

$$R_s^{d_{\rm B}} \propto s. \tag{12}$$

Comparing these equations, we get $G(x) \propto x^{1/d_{\rm B}}$ and the hyperscaling relation

$$d_{\rm B}\nu = 1/\sigma_{\rm D}. \tag{13}$$

Equations (7), (10) and (13) give expressions of all critical exponents for dead-end distribution in terms of ν, β (d_C) and β_B (d_B)

$$2-\alpha_D = d\nu-\beta = d_C\nu, \tag{14}$$

$$\beta_D = \beta_B-\beta = (d_C-d_B)\nu, \tag{15}$$

$$\gamma_D = d\nu+\beta-2\beta_B = (2d_B-d_C)\nu, \tag{16}$$

$$\delta_D = (d\nu-\beta_B)/(\beta_B-\beta) = d_B/(d_C-d_B), \tag{17}$$

$$\sigma_D = 1/(d\nu-\beta_B) = 1/d_B\nu, \tag{18}$$

$$\tau_D = 1+(d\nu-\beta)/(d\nu-\beta_B) = 1+d_C/d_B. \tag{19}$$

Using known estimates for ν, β and β_B (Stauffer 1979, Kirkpatrick 1978, Stanley 1977), we can evaluate explicit values of critical exponents. The results are listed in table 1. These critical exponents may provide useful information on physical properties of percolating systems.

Table 1. Critical exponents for dead end distribution

d	ν^a	β^a	β_B^b	α_D	β_D	γ_D	δ_D	σ_D	τ_D
1	1	0	0	1	0	1	∞	1	2
2	1.35	0.14	0.5	−0.6	0.4	1.8	6.1	0.5	2.2
3	0.84	0.4	0.9	−0.1	0.5	1.1	3.2	0.6	2.3
4	0.7	0.5	1.1	−0.3	0.6	1.1	2.8	0.6	2.4
5	0.6	0.7		−0.3					
6	0.5	1	2^c	0	1	0	1	1	3

[a] Stauffer (1979), [b] Kirkpatrick (1978), [c] Stanley (1977).

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